New Existence Results for Efficient Points in Locally Convex Spaces Ordered by Supernormal Cones

VASILE POSTOLICĂ
Department of Mathematics, B-dul Traian nr. 11, bl. A_1, sc. A, ap. 13, 5600 – Piatra Neamç, Romania

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Abstract. In this paper, the definition of supernormality for convex cones in locally convex spaces is discussed in detail on many interesting examples. Starting from the new direction for the study of the existence of efficient points (Pareto type optimums) in locally convex spaces offered by the concept of supernormal (nuclear) cone, we establish some existence results for the efficient points using boundedness and completeness of conical sections induced by non-empty subsets and we specify properties for the sets of efficient points beside important remarks

Key words. Supernormal cone, well based cone, H-locally convex space, efficient point, cone-bounded (cone closed) set.

1. Introduction

It is well known that in the decision or game theories the vectorial optimization is more appropriate than the strong optimization. Moreover, the vectorial optimization programs are very useful in multiobjective control problems for Pareto type optimality. These are the reasons to present in this paper which is in connection with the main directions of research concerning with the vectorial optimization (existence conditions for optimums and duality, the study of properties for sets of efficient points and the elaboration of numerical methods especially for nonlinear vectorial optimization problems) suggestive examples for supernormal cones with an examination of supernormality in H-locally convex spaces, existence results for efficient points in locally convex spaces ordered by cones such as these, properties of sets of efficient points and related important remarks. At the same time, the following theorems show that the supernormality is a reasonable restriction on convex cones in order to avoid the compactness assumption imposed usually upon the objective sets because it together with proper conditions on conical (extensions) sections of non-empty sets ensure the existence for the efficient points in locally convex spaces. Thus, the use of supernormal cones for the study of the existence of efficient points is a new direction for the investigations of Pareto type optimums in infinite dimensional spaces, especially in locally convex spaces, which, in general, have unknown geometry (and, consequently, there are not geometrical interpretations for the structure of efficient points sets) since it is

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based on completeness instead of compactness or of the usual assumption that the cones have non-empty interiors.

All the elements of ordered topological vector spaces used in this work are in accordance with [14].

2. Supernormal Cones and Related Topics

Let X be a Hausdorff locally convex spaces with the topology induced by a family $\mathcal{P} = \{p_\alpha : \alpha \in I\}$ of seminorms, ordered by a convex cone K and its topological dual space X'. Before we give the main examples, comments and results of this section, we recall a basic definition and an important theorem.

DEFINITION 2.1. [7]. K is said to be supernormal (nuclear) if for every $p_\alpha \in \mathcal{P}$, there exists $f_\alpha \in X'$ so that $p_\alpha(x) \leq f_\alpha(x)$ for all $x \in K$.

THEOREM 2.1. [1]. *In a Hausdorff locally convex space a convex and normal cone K is supernormal if and only if every generalized sequence of K weakly convergent to zero converges to zero in the locally convex topology.*

EXAMPLES.

1. Any convex, closed and pointed cone in an arbitrary usual Euclidean space R^k is supernormal.
2. In every locally convex space any well based convex cone (i.e., generated by a convex, bounded set which does not contain the origin in the closure) is supernormal.
3. In a normed space, a convex cone is supernormal if and only if it is well based.
4. In a locally convex space, every locally compact (weakly locally compact) convex cone is supernormal.
5. In a nuclear space [13] a convex cone is supernormal iff it is normal.
6. In a locally convex space a convex cone is weakly supernormal if and only if is weakly normal.
7. In $L^p([a, b])$ $(p \geq 1)$ the convex cone $K_p = \{x \in L^p([a, b])\colon X(t) \geq 0$ almost everywhere$\}$ is supernormal if and only if $p = 1$, being well based in this case by set

$$B = \left\{x \in K_1 \colon \int_a^b x(t)\, \mathrm{d}t = 1\right\}.$$

Indeed, if $p > 1$, then the sequence (x_n) defined by

$$x_n(t) = \begin{cases} n^{1/p}, & a \leq t \leq a + \dfrac{b-a}{2n} \\ 0, & a + \dfrac{b-a}{2n} < t \leq b \end{cases}, \quad n \in N^*$$

converges to zero in the weak topology but not in the usual norm topology. Therefore, by virtue of Theorem 2.1, K_p is not supernormal. A similar result holds for $L^p(R)$. Thus, if we consider a countable family (A_n) of disjoint sets which covers R such that $\mu(A_n) = 1$, for all n in N, where μ is the Lebesgue measure, then the sequence (y_n) given by $y_n(t) = 1$ if $t \in A_n$ and $y_n(t) = 0$ for $t \in R \backslash A_n$ converges weakly to zero while it is not convergent to zero in the norm topology. Taking into account the above theorem, it follows that the usual cone in $L^p(R)$ is not supernormal if $p > 1$, that is, it is not well based in these cases. However, these cones are normal for every $p \geq 1$.

8. In l^p (p ≥ 1) equipped with the usual norm $\|\cdot\| p$ the positive cone $C_p = \{(x_n) \in l^p\colon x_n \geq 0 \text{ for all } n \in N\}$ is also normal with respect to the usual norm topology but it is not supernormal excepting the case $p = 1$. Indeed, for every $p > 1$, the sequence (e_n) having 1 on the nth coordinate and zeros elsewhere converges to zero in the weak topology but not in the norm topology and by virtue of Theorem 2.1 it follows that C_p is not supernormal. For $p = 1$, C_p is well based by the set $B = \{x \in C_1\colon \|x\|_1 = 1\}$ and by Proposition 5 of [8] it is supernormal. If we consider in this case the locally convex topology in l^1 defined by the seminorms

$$p_n((x_k) = \sum_{k=0}^{n} |x_k| \text{ for every } (x_k) \text{ } in \text{ } l^1 \text{ and } n \in N^*,$$

which is weaker than its usual weak topology, then the usual positive cone remains supernormal with respect to this topology (now it is normal in a nuclear space and we apply Proposition 6 of [8]) but it is not well based (see also Example IV.3.3 given in Chapter 4 of [1]). Taking into account the concept of H-locally convex space introduced by T. Precupanu in [17] and defined as a Hausdorff locally convex space with the seminorms satisfying the parallelogram law and the property that every nuclear space is also a H-locally convex space with respect to an equivalent system of seminorms [13], the above example shows that even in a H-locally convex space a proper convex cone may be supernormal without to be well based. Moreover, if we consider in l^2 the H-locally convex topology induced by the seminorms

$$\tilde{p}_n((x_k)) = \left(\sum_{i \geq n} |x_i|^2\right)^{1/2}, \quad n \in N^*, \quad (x_k) \in l^2$$

then the convex cone $C_2 = \{(x_k) \in l^2\colon x_k \geq 0 \text{ for all } k \in N\}$ is normal in the H-locally convex space $(l^2, \{\tilde{p}_n\}_{n \in N^*})$, but it is not supernormal because the same sequence (e_k) is weakly convergent to zero, while $(\tilde{p}_n(e_k))$ is convergent to 1 for each $n \in N^*$ and one applies Theorem 2.1. Another interesting example of normal cone in a H-locally convex space which is not supernormal is the usual positive cone in the space $L^2_{\text{loc}}(R)$ of all functions from R to C which are square integrable over any finite interval of R with the system of seminorms $\{\bar{p}_n\colon n \in N^*\}$ defined by $\bar{p}_n(x) = (\int_{-n}^{n} |x(t)|^2 \, dt)^{1/2}$, for every x in $L^2_{\mathrm{loc}}(R)$. In this case, the sequence (x_k) given by

$$x_k(t) = \begin{cases} 0, & t \in (-\infty, 0) \cup \left(\frac{1}{k}, +\infty\right) \\ \sqrt{k}, & t \in \left[0, \frac{1}{k}\right] \end{cases}$$

converges weakly to zero but it is not convergent to zero in the H-locally convex topology. The result follows again by Theorem 2.1. These examples show that, in comparison with the normed spaces or the nuclear spaces, the only advantage offered by H-locally convex spaces is that we know the expression of linear and continuous functionals in H-locally convex spaces which are Fréchet spaces [10] if we want to study the supernormality in such spaces as these (the weak topologies are H-locally convex but, in these cases, the supernormality coincides with the normality by Corollary of Proposition 2 in [8]). We find a characterization of supernormality in H-Fréchet spaces in the following Theorem 2.3.

9. In the space $C([a, b])$ of all continuous real valued functions on a compact interval $[a, b]$ equipped with the usual supremum norm the convex cone $K = \{x \in C([a, b])$: x is concave, $x(a) = x(b) = 0$ and $x(t) \geq 0$ for all $t \in [a, b]\}$ is supernormal, being well based by every set $\{x \in K\colon x(t_0) = 1\}$ for some arbitrary $t_0 \in [a, b]$. The hypothesis that all $x \in K$ are concave is essential for supernormality.
10. The convex cone of all nonnegative sequences in the space of all absolutely convergent sequences is the dual of the usual positive cone in the space of all convergent sequences. Consequently, it has a weak star compact base and hence it is weak star supernormal.
11. In l^∞ or in c_0 equipped with the supremum norm, the convex cone consisting of all sequences having all partial sums non-negative is not normal, hence it is not supernormal.

REMARK 2.1. The concept of supernormal cone was defined by G. Isac in [7] and the importance of the supernormal cones for the existence of the solutions for vectorial optimization problems in locally convex spaces was emphasized for the first time in [8] where we also find examples 1–6. As a consequence of the scientific cooperation with Professor G. Isac, we completed the list of examples given in [7], [8] and so on with other interesting examples and comments.

REMARK 2.2. It is clear that every supernormal cone is pointed and that closure of any supernormal cone is also supernormal. On the other hand, if $(V, \|\cdot\|)$ is a normed linear space, then the fact that a convex cone C is supernormal in V implies $C \subseteq \{v \in V\colon \|v\| \leq f(v)\}$ for some linear and continuous functional f of V, which involves the existence of C-strictly positive linear functionals. The space $B([a, b])$ of all bounded real valued functions on an interval $[a, b]$ with the usual norm and its standard positive cone $C = u \in B([a, b])$: $u(t) \geq 0$ for all $t \in [a, b]\}$ is an example of nonexistence of C-strictly positive functionals (see the second example given in [14], p. 27). Therefore, this is another example of normal cone in a Banach space which does not have a base, that is, which is not supernormal. Moreover, it is possible even for a normal cone to admit strictly positive linear continuous functionals without to be supernormal. This is the case of the usual positive cone in $L^p([a, b])$ when $p > 1$.

LEMMA 2.1. *Every hyperplane which separates strictly the base of a well based cone and the origin of the space is not contained in the cone.*

Proof. Let $K = \bigcup_{n \geq 0} \lambda B$ be a convex cone well based by a bounded convex set B with the property that the origin is not in its closure and $H = \{x \in X$: $f(x) = c\}$ a hyperplane which separates strictly B and the origin of the space X. Then, the set $M = \{x \in K$: $f(x)/c = 1\}$ is a base for K (in general, this base is not bounded; it is bounded if X is a reflexive Banach space). By virtue of Proposition 4 given in [9], the intersection between M and every convex cone having compact base and the apex in an arbitrary point, in particular, with any straight line is a bounded set. Thus, if we assume that H is contained in K, then all the straight lines of $(1/c)H$ are bounded, a contradiction. The result follows.

THEOREM 2.2. *There exists no maximal well based convex cone with respect to the inclusion relation.*

Proof. Let K be a well based convex cone in X, that is, there exists a bounded, convex set B with its closure $\bar{B}$ such that $0 \notin \bar{B}$ and $K = \bigcup_{\lambda \geq 0} \lambda B$. By the Classical Separation Theorem in locally convex spaces, there exists a linear and continuous functional f, $\epsilon > 0$ and $c \in R$ such that the hyperplane $H = \{x \in X$: $f(x) = c\}$ separates strictly $\{0\}$ and $\bar{B}$, that is, $f(x) \leq c - \epsilon$, $x \in \bar{B}$ and $0 \geq c + \epsilon$. Taking into account Lemma 2.1 it follows that there exists $t_0 \in H \cap (X \setminus K)$. Then, the closure B_1 of the convex hull of the set $\bar{B} \cup \{t_0\}$ is bounded, convex, does not contain the origin and generates the cone K_1 which contains K. This completes the proof.

REMARK 2.3. The proof of Lemma 2.1 was suggested by Professor G. Isac, Département de Mathématiques, Collège Militaire Royal St-Jean Québec, Canada.

If X is a H-Fréchet space, that is, the family $\mathcal{P}$ is countable and every seminorm $p_\alpha(\cdot)$ is generated by the scalar semi-product $(\cdot \cdot \cdot)_\alpha$, $\alpha \in I$, then taking into account Definition 2.1, Theorem 2.1 of [10] and the results of [5] we obtain the following characterization of supernormality through the agency of ϵ-subdifferential:

THEOREM 2.3. *A convex cone K is supernormal in the H-Fréchet space $(X, \mathcal{P})$ if and only if for every seminorm $p_\alpha(\cdot)$, there exists $y_\alpha \in X$ such that the*

ϵ-subdifferential of $p_\alpha(\cdot)$ at the origin of the space is contained in the translation of the polar cone to K by the linear and continuous functional $(\cdot, y_\alpha)_\beta$ for some index β in I, whenever $\epsilon \geqslant 0$.

3. Existence Results for Efficient Points

In the conditions of the preceding section, let A be a non-empty subset of X and $\bar{x} \in A$.

DEFINITION 3.1. We say that $\bar{x}$ is an efficient point for A with respect to K, in notation, $\bar{x} \in \mathrm{MIN}_K(A)$, if $A \cap (\bar{x} - K) = \{\bar{x}\}$.

The first existence result for the efficient points is based on supernormality of K, the boundedness and completeness of conical (extension) sections induced by non-empty subsets and the following main theorem of [8].

THEOREM 3.1. [8]. *If K is a supernormal cone in a Hausdorff locally convex space E and S is a non-empty subset of E having the property that there exists a bounded and complete set $S_0 \subseteq S$ with $S \cap (K + x) \subseteq S_0$ for every $x \in S_0$, then there exists $x_0 \in S$ such that $s \cap (K + x_0) = \{x_0\}$.*

REMARK 3.1. The proof of this theorem given in [8] shows in fact that there exists at least a critical point for the generalized dynamical system $\Gamma: S_0 \to 2^{S_0}$ defined by $\Gamma(x) = S \cap (K + x)$ and hence it suggests the implications of supernormal cones for the equilibrium theory in infinite dimensional spaces.

THEOREM 3.2. *Let $A \subseteq B \subseteq A + K$. If K is supernormal and $B \cap (A_0 - K)$ is bounded and complete for some non-empty set $A_0 \subseteq A$, then* $\mathrm{MIN}_K(A) \neq \emptyset$

Proof. Let $A' = B \cap (A_0 - K)$ with $A_0 \subseteq A$ such that A' is bounded and complete. Since $A' \cap (a' - K) \subseteq A'$ for every $a' \in A'$ by virtue of Theorem 3.1 it follows that $\mathrm{MIN}_K(A') \neq \emptyset$. But $\mathrm{MIN}_K(A')$ is contained in $\mathrm{MIN}_K(A)$. Indeed, if $x \in \mathrm{MIN}_K(A')$ and we assume that $x \not\subseteq A$ then there exist $a \in A$ and k in $K \backslash \{0\}$ such that $x = a + k$. On the other hand, $x = a_0 - k_1$ with $a_0 \in A_0$ and $k_1 \in K$, therefore $a = x - k \triangleq a_0 - (k + k_1)$. Consequently, $a \in A'$ and $x - a \in K \backslash \{0\}$, a contradiction. Hence $\mathrm{MIN}_K(A') \subseteq A$.

Suppose now that there exists $x \in \mathrm{MIN}_K(A') \backslash \mathrm{MIN}_K(A)$. Then, there exists $a_1 \in A$ such that $x - a_1 \in K \backslash \{0\}$. Therefore $a_1 \in x - K \subseteq A_0 - K$ and $a_1 \in A \subseteq B$, that is, $a_1 \in A'$, a contradiction. Consequently, $\mathrm{MIN}_K(A') \neq \emptyset$ and $\mathrm{MIN}_K(A') \subseteq \mathrm{MIN}_K(A)$.

REMARK 3.2. The proof of the above theorem shows that if K is supernormal and $A \cap (a - K)$ or $(A + K) \cap (a - K)$ is bounded and complete for some $s \in A$, then $\mathrm{MIN}_K(A) \neq \emptyset$. When this boundedness and completeness property holds for every $a \in A$, that is, every section or conical section of A is bounded and complete, then $A \subseteq \mathrm{MIN}_K(A) + K$. This inclusion is very useful to establish properties for the sets of efficient points (Theorem 3.3) and for duality theory in vectorial optimization programs with objective maps multifunctions [16]. Also we must remark that, in general, the uniqueness of the efficient points is unusual. Excepting the trivial cases, it seems to hold only in strong optimization problems under proper conditions.

COROLLARY 3.2.1. *If A is a non-empty, bounded and closed subset of X and K is well based by a complete set, then* $\mathrm{MIN}_K(A) \neq \emptyset$ *and* $A \subseteq \mathrm{MIN}_K(A) + K$.

Proof. Since A is bounded and K is supernormal (Proposition 5 of [8]), by Theorem 3.2 it is sufficient to prove that every section of A with respect to K is complete. Let $a \in A$ be an arbitrary element and let $(a_j)_{j \in J}$ be a Cauchy net in $A \cap (a - K)$. Because K is well based by a complete set, there exists a non-empty, convex, bounded and complete set B such that $0 \notin B$ and $K = \bigcup_{\lambda \geq 0} \lambda B$. Hence, for each a_j $(j \in J)$, there exist $\lambda_j \geq 0$ and $b_j \in B$ with $a_j = a - \lambda_j b_j$. Therefore, $(\lambda_j b_j)_{j \in J}$ is a bounded Cauchy net. Since the set B is closed, bounded and $0 \notin B$, there exists a convex and closed neighbourhood V of the zero element in X and $\alpha > 0$ such that $V \cap B = \emptyset$ and $B \subseteq \alpha V$. If p_v is the Minkowski functional of V, then $1 \leq p_v(b) \leq \alpha$ for every $b \in B$ and there exists $M \geq 0$ with $\lambda_j \leq p_v(\lambda_j b_j) \leq M$ for all λ_j, that is, $(\lambda_j)_{j \in J}$ is bounded. When $(\lambda_j)_{j \in J}$ contains at least a subnet convergent to zero, then it is clear that a_j tends to a; otherwise, because it is bounded, we can find a subnet $(\lambda_s)_{s \in S}$ convergent to $\lambda_0 > 0$. Since $(a_s)_{s \in S}$ is a Cauchy net, $(b_s)_{s \in S}$ is a Cauchy net on B. Therefore $(b_s)_{s \in S}$ converges to $b_0 \in B$ and $(s_s)_{s \in S}$ is convergent to $a - \lambda_0 b_0$ which implies that $(a_j)_{j \in J}$ converges to $a - \lambda_0 b_0$.

REMARK 3.3. Because in a Hausdorff locally convex space a pointed cone is locally compact if and only if it has a compact generating base, the conclusion of the above corollary remains valid whenever the cone K is closed and locally compact.

REMARK 3.4. If K is normal and $A \subset X$ is bounded, then $[A] = (A + K) \cap (A - K)$ is bounded. Indeed, let ϑ be a neighbourhood basis of the origin in X such that $[V] = V$, $\forall V \in \vartheta$. Because A is bounded, for every $V \in \vartheta$, there exists $\lambda > 0$ such that $A \subseteq \lambda V$, which implies $A + K \subseteq \lambda V + K = \lambda(V + K)$ and $A - K \subseteq \lambda V - K = \lambda(V - K)$, that is, $[A] \subseteq \lambda(V + K) \cap \lambda(V - K) = \lambda[V] - \lambda V$ and the result follows.

DEFINITION 3.2. A non-empty set $B \subset X$ is K-bounded if there exists a bounded set $B_0 \subset X$ such that $B \subseteq B_0 + K$ and B is said to be K-closed if its conical extension $B + K$ is closed.

REMARK 3.5. For all we know compactness is the strongest demand on a given non-empty set concerning with the existence of the efficient points, so if we want to obtain existence results in a less restrictive class of non-empty sets, we must impose adequate conditions on the cone. Also the results of this paper show that such a hypothesis as this is the supernormality. It is clear that for a non-empty set to be compact is more restrictive than to be K-bounded and K-closed. Several properties of cone-bounded and cone-closed sets in infinite dimensional spaces we find in [12].

In the following theorem we assume that X is quasi-complete that is, every non-empty, bounded and closed subset is complete, and K is closed and supernormal.

THEOREM 3.3. (i) *For every non-empty K-bounded and K-closed subset A we have* $\mathrm{MIN}_K(A) \neq \emptyset$ *and* $A \subseteq \mathrm{MIN}_K(A) + K$;

(ii) *if the set* $B \cap (A_0 - K)$ *is K-bounded and K-closed for some non-empty sets B and* A_0 *with* $A \subseteq B \subseteq A + K$ *and* $A_0 \subseteq A$, *then* $\mathrm{MIN}_K(A) \neq 0$;

(iii) *for every K-bounded and K-closed set* $A \subset X$, $\mathrm{MIN}_K(A) + K = A + K$ *and* $\mathrm{MIN}_K(A)$ *is K-bounded and K-closed*;

(iv) *for every K-bounded and K-closed subsets A, B we have*:

$$\mathrm{MIN}_K(A+B) \subseteq \mathrm{MIN}_K(A) + \mathrm{MIN}_K(B), \mathrm{MIN}_K(A+B) = \mathrm{MIN}_K[\mathrm{MIN}_K(A) + B]$$
$$= \mathrm{MIN}_K[A + \mathrm{MIN}_K(B)]$$
$$= \mathrm{MIN}_K[\mathrm{MIN}_K(A) + \mathrm{MIN}_K(B)] \text{ and } \mathrm{MIN}_K(A \cup B)$$
$$= \mathrm{MIN}_K[\mathrm{MIN}_K(A) \cup \mathrm{MIN}_K(B)]; \text{ if, in addition, } A \subseteq B, \text{ then}$$
$$\mathrm{MIN}_K(A) \subseteq \mathrm{MIN}_K(B) + K;$$

(v) *if* K_1, K_2 *are two closed and supernormal cones in X and A is a* K_1, K_2*-bounded and* K_1, K_2*-closed subset, then*

$$\mathrm{MIN}_{K_1}(A) \cap \mathrm{MIN}_{K_2}(A) = \mathrm{MIN}_{\mathrm{cone}(K_1 \cup K_2)}(A) .$$

Proof. (i) In the conditions of theorem, every conical extension section of A is bounded and closed and the result follows by Theorem 3.2 and Remark 3.2. (ii) is a consequence of (i). (iii)-(v) are based on the inclusion $A \subseteq \mathrm{MIN}_K(A) + K$ for every K-bounded and K-closed subset A.

REMARK 3.6. Simple examples show that the above inclusion is not valid, in general, even if $\mathrm{MIN}_K(A)$ is non-empty and X is an usual Euclidean space. On the other hand, even when X is quasi-complete and A is K-bounded and K-closed the equality $A = \text{MIN}_K(A) + K$ is equivalent with $A = A + K$ and hence it holds in some special cases which depend only upon the structure of the set A and the cone K. For example, if A is bounded and K-closed, then the equality $A = \text{MIN}_K(A) + K$ is impossible because $A + K$ is, generally, unbounded. Therefore, even in these cases, the inclusion $A \subseteq \text{MIN}_K(A) + K$ will be strict. Other existence results for the efficient points and connections with the conically bounded sets may be found in [15]. Also we must to specify that the notion of conically bounded set does not coincide, in general, with the concept given by Definition 3.2 of this paper. It was considered for the first time in Banach spaces by Bourgin [3] with applications in the study of reflexivity and in locally convex spaces by Isac [9] to examine the global minimization of a nonlinear functional on a convex cone.

REMARK 3.7. Generally, the study of solutions for vectorial optimization problems, in particular, of the efficient points has been made using Zorn's lemma in order to establish existence theorems (see, for instance, [2], [4], [11]), by reducing them to scalar optimization problems and afterwards one applies known results from scalar optimization theory ([2], [6], [11] and other connected papers) or by the aid of the duality theory for vectorial optimization programs with objective maps multifunctions [16]. In this paper our approach is much different because it does not depend upon the finite dimensionality of the space, the compactness of the set for which we search the efficient points or the usual restriction that the cone has non-empty (relative) interior.

REMARK 3.8. In [18] was defined the largest class $\mathscr{C}$ of convex cones ensuring the existence of the efficient points in compact sets: if V is a Hausdorff topological vector space, a convex cone C belongs to $\mathscr{C}$ when for every closed vector subspace L of V, $C \cap L$ is a vector subspace whenever its closure $\overline{C \cap L}$ is a vector subspace. From Theorem 3.2 together with the maximality property of $\mathscr{C}$ or by the considerations below, it follows that in every Hausdorff locally convex space any supernormal cone belongs to $\mathscr{C}$. Indeed, let V be an arbitrary Hausdorff locally convex space with the topology generated by the family $\mathscr{Q} = \{q_s : s \in S\}$ of seminorms and $C \subset V$ a supernormal cone. The definition of supernormality does not include the assumption that the cone C is closed. It is clear that if we assume that C is closed, then automatically $C \in \mathscr{C}$. If we do not suppose that C is closed, then the answer is the same, but it requires some calculations: for every $q_s \in \mathscr{Q}$ let us denote the linear and continuous functional given by the definition of supernormality with f_s. Then, we have $C \subseteq \bigcap_{q_s \in \mathscr{Q}} \{v \in V : q_s(v) \leq f_s(v)\} = C_1$. Since q_s and $f_s (s \in S)$ are continuous, we conclude that C_1 is a closed, convex cone and therefore $\overline{C} \subseteq C_1$. But C_1 is also pointed because if $x, -x \in C_1$, then $0 \leq q_s(x) + q_s(-x) \leq f_s(x) + f_s(-x) = 0$ for all $q_s \in \mathscr{Q}$. Since V is a Hausdorff topological vector space, we must have $x = 0$. Hence C is a convex cone whose closure is pointed and by virtue of Remark 2.2 given in [18] it follows that $C \in \mathscr{C}$.

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